Vertical Motion of a Buoy-Cable-Array System Used in Submarine Detection

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The vertical motion of a submarine detection system consisting of a surface float, cable, and an array of three neutrally buoyant inflated cylindrical legs is investigated. The variation of the natural frequencies of free vibration with different design parameters is studied. Analysis of the response of the system to surface wave excitations is carried out, the final objective being the reduction in the displacements of the leg tips supporting the hydrophones.

Nomenclature

B_{ij} = coefficients in the eigenvalue expansion of w_i, Equation (1b)
C_d, C_{db}, C_{dh} = drag coefficients of the leg, buoy, and head, respectively
C_m, C_{mb}, C_{mh} = added inertia coefficients of the leg, buoy, and head, respectively
C_{sj} = coefficients in the eigenfunction expansion of $d^2\Phi_s/d\xi^2$, Eq. (8c)
E = Young's modulus
$E^*(\omega)$ = complex modulus
F_a = axial tension
F_b, F_h, F_i = total hydrodynamic forces on the buoy, head and ith leg, respectively; $i = 1,2,3$
$\bar{F}_{ia}$ = axial force acting on an element of the ith leg; $i = 1,2,3$
H = depth of the central head below the water surface
I = moment of inertia of the cross section of a leg
I_i = $2\pi(i-1)/3$; $i = 1,2,3$
L = length of the leg
P = pressure parameter, Eq. (10)
$\bar{P}$ = weighted pressure parameter, Eq. (10)
Q_k = nonconservative generalized forces corresponding to the generalized coordinates q_k $(q_k \equiv B_{ij}, z_b, z_h)$
Q'_k = contribution of the follower forces to Q_k
Q''_k = contribution of the hydrodynamic forces to Q_k
S, S_b, S_h = areas of cross section of the leg, buoy, and head, respectively
T = kinetic energy
$\bar{T}$ = period of the wave
U = potential energy
a, a_b, a_h = added inertia of the leg, buoy, and head, respectively
b_{ij} = coefficients in the eigenfunction expansion of ζ_i
c = equivalent stiffness due to the buoyancy
d = diameter of each leg
f, f_b, f_h = coefficients of forcing functions in the vertical motion of the system, Eq. (10)
g = acceleration due to gravity
h = wall thickness of each leg
$\bar{i}, \bar{j}, \bar{k}$ = unit vectors along x_0, y_0, z_0 axes, respectively
k = equivalent spring stiffness of the cable
m, m_b, m_h = mass of each leg, the buoy, and head, respectively
p = internal pressure
q_k = generalized coordinate
$r_{h\ell}, r_{b\ell}$ = inertia parameters, Eq. (10)
t = time
w_i = flexural displacements of an element of the ith leg
x_0, y_0, z_0 = inertial coordinates axes
z_b, z_h = vertical displacements of the buoy and head, respectively
z_w, z_{wh}, z_{wi} = displacement of a water particle at the buoy, central head and a point on the ith leg, due to the ocean waves, respectively
$\Phi_j(\xi)$ = eigenfunctions of a cantilever without axial force, Eq. (1c)
Ω_j = dimensionless jth natural frequency of each leg, $j = 1,2,\ldots,\infty$
$\bar{\Omega}$ = square root of the ratio of the stiffness of the spring to that due to the buoyancy $(k/c)^{1/2}$
$\alpha, \alpha_b, \alpha_h$ = damping parameter of each leg, the buoy and head, respectively, Eq. (10)
δ_j = constant, $2\sigma_j/\mu_j$; $j = 1,2,\ldots,\infty$
η_0 = amplitude of η_w
η_b, η_h = dimensionless vertical displacements of the buoy and head, respectively
$\eta_s, \eta_c, \eta_{bs}, \eta_{bc}, \eta_{hs}, \eta_{hc}$ = sine and cosine components of η, η_b, and η_h, respectively
η_i = dimensionless flexural displacement of an element of the ith leg
$\eta_w, \eta_{wh}, \eta_{wi}$ = dimensionless displacements of a water particle at the buoy, central head and a point on the ith leg, due to the ocean waves, respectively
λ_j = jth eigenvalue
μ_j = eigenvalues of a cantilever
ξ = dimensionless distance from the fixed end of a cantilever
ρ_w = density of water
σ_j = $(\cosh\mu_j + \cos\mu_j)/(\sinh\mu_j + \sin\mu_j)$
τ = dimensionless time
ω = frequency
$(\dot{\ })$ = differentiation with respect to t
$(\ )'$ = differentiation with respect to τ

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I. Introduction

NEUTRALLY buoyant inflated structures have been proposed for a variety of missions because of their compactness and light weight. Consider, for example, the problem of patrolling of submarines. It is currently undertaken in various ways, such as: 1) long range patrol aircraft equipped with radar which can detect the surfaced or snorkeling subs;

2) turnstiles placed across the various gateways to the major ocean basins; 3) fixed site or towed sonar systems; 4) sonobuoys providing platforms for hydrophones and telemetering systems; etc. Of particular interest is the last option. Sonobuoys are passive listening devices housed usually in a cylindrical container about 3 ft long, and 5 to 6 in. in diam. The containers are dropped from an aircraft in the area of interest. Upon hitting the water surface, a hydrophone attached by a cable to the floating container is released. The system transmits all the signals received by the hydrophone back to the aircraft. Theoretically, at least three or four hydrophones are needed to locate an object in two or three dimensions, respectively.

The sonobuoy has a certain lifetime, after which it ceases to function and is allowed to sink. It has been established that the efficiency of this operation can be improved considerably by use of an array of inflatable tubes, each carrying a hydrophone at one end and joined to a central head, equipped with a pump, at the other (Fig. 1). The pump pressurizes the tubes with water, making them neutrally buoyant. An object then can be located through processing of signals received by the array, provided the position and orientation of the array are known.

Since the system, under normal operating conditions, will be subjected to the ocean currents, waves, and other local disturbances, the knowledge of its dynamics is of fundamental importance for evolving suitable design procedures. The objective of this paper is to study the vertical motion of such a buoy-cable-array assembly in the presence of surface wave excitation.

The nylon cable used for the suspension of the system has a very small stiffness in the axial direction as compared to the other modes of cable motion. Hence, in this analysis, the cable is replaced by a spring of equivalent stretching stiffness such that the system reduces to a buoy and an array connected by a spring. The central head of the array is allowed to move vertically and the flexural displacements of the legs are superposed on this motion. To begin, a general formulation of the problem is presented, using the classical Lagrangian procedure. The free vibration of the system is considered first by equating the forcing terms in the equations of motion to zero. The influence of the important system parameters on the natural frequencies of vertical motion is evaluated. Subsequently, the motion excited by a sinusoidal surface wave is investigated. Attempts are made to determine the effects of various parameters on the tip displacements at the locations of the hydrophones.

II. Formulation of the Problem

Consider a system comprised of a cylindrical surface float connected by an elastic cable to a central head, supporting three neutrally buoyant inflated cylindrical cantilevers (Fig. 2). Let m_b and m_h be the masses of the buoy and central head, respectively, and L and d be the length and diameter of each leg. The cable is replaced by an equivalent spring of stiffness k. An inertial coordinate system x_0,y_0,z_0 is located at the free surface as shown in Fig. 2 such that $(0,0,z_{b0})$ and $(0,0,-H)$ are the coordinates of the equilibrium positions of the centers of mass of the buoy and central head, respectively. At any instant t, the locations of the centers of mass of the buoy and the head and a point on the ith leg, at a distance $\xi L\,(0\leq\xi\leq 1)$ from the root, are given by $(0,0,z_{b0}+z_b)$, $(0,0,-H+z_h)$, and $(\xi L\cos I_i,\ \xi L\sin I_i,\ -H+z_h+w_i)$, respectively, where

$$I_i=2\pi(i-1)/3 \tag{1a}$$

The flexural displacements w_i can be expanded in series form

$$w_i=\sum_{j=1}^{\infty}\Phi_j(\xi)B_{ij}(t) \tag{1b}$$

where the set of orthonormal functions $\Phi_j(\xi)$ satisfying the

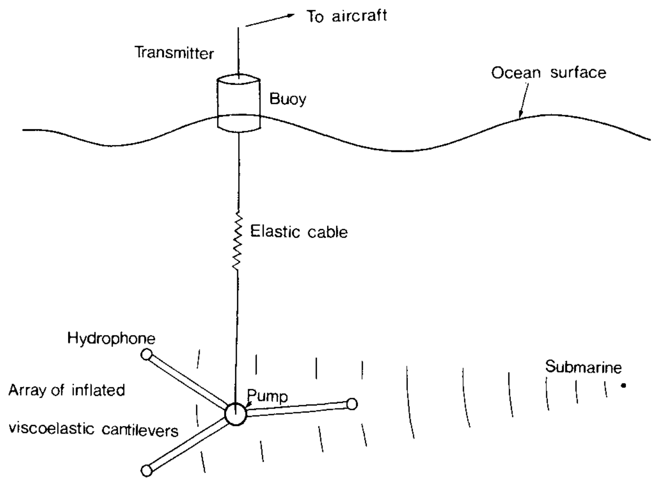

Fig. 1 Schematic diagram of a submarine detection system using an array of inflated structural members.

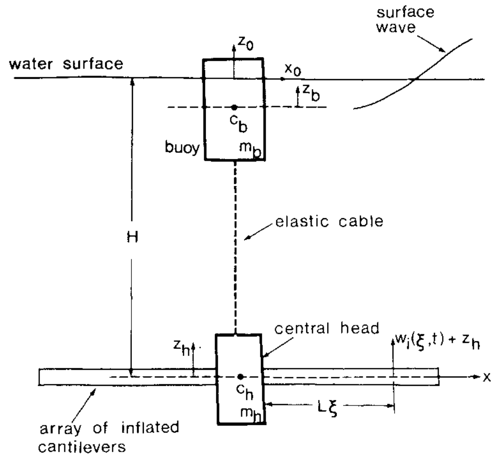

Fig. 2 Geometry of vertical motion of the buoy-cable-array assembly.

boundary conditions for the leg are given by

$$\Phi_j(\xi)=(\cosh\mu_j\xi-\cos\mu_j\xi)-(\cosh\mu_j+\cos\mu_j)(\sinh\mu_j\xi-\sin\mu_j\xi)/(\sinh\mu_j+\sin\mu_j) \tag{1c}$$

μ_j being the eigenvalues for a cantilever. It may be noted that, even in the presence of axial follower forces, the boundary conditions are the same as those in the case of a cantilever.[1]

The kinetic energy T of the system is given by

$$T=\left(\frac{m_b}{2}\right)\dot z_b^2+\left(\frac{m_h}{2}\right)\dot z_h^2+\left(\frac{1}{2}\right)\sum_{i=1}^{3}\int_m\Big\{\dot z_h+\sum_{j=1}^{\infty}\dot B_{ij}\Phi_j\Big\}^2dm=\left(\frac{m_b}{2}\right)\dot z_b^2+\left(\frac{m_h}{2}\right)\dot z_h^2+\left(\frac{m}{2}\right)\Big[3\dot z_h^2+\sum_{i=1}^{3}\sum_{j=1}^{\infty}\dot B_{ij}^2+2\dot z_h\delta_j\dot B_{ij}\Big] \tag{2}$$

where m is the mass of each cylinder, including the water inside it, and δ_j is defined by

$$\delta_j=(2/\mu_j)(\cosh\mu_j+\cos\mu_j)/(\sinh\mu_j+\sin\mu_j)$$

This does not include the kinetic energy associated with the apparent inertia of the assembly, since the effect would be accounted for in the generalized forces.

The potential energy U of the system consists of three parts: the energy associated with the buoyancy of the buoy, the elastic energy stored in the cable, and that due to the flexural displacements of the legs. The nonconservative follower forces arising because of the internal pressure do not contribute to the potential energy. Hence,

$$U=\left(\frac{c}{2}\right)(z_b-z_w)^2+\left(\frac{k}{2}\right)(z_b-z_h)^2+\left(\frac{EI}{2L^3}\right)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\mu_j^4B_{ij}^2 \quad (3)$$

where c is the equivalent stiffness due to buoyancy, and can be written as

$$c=(\rho_w g)\ \text{(area of the cross section of the buoy)}$$

and z_w is the displacement of a water particle due to the wave at the free surface. Using Eqs. (2) and (3), the classical Lagrangian formulation yields

$$m_b\ddot{z}_b+c(z_b-z_w)+k(z_b-z_h)=Q_{zb} \quad (4a)$$

$$(m_h+3m)\ddot{z}_h+k(z_h-z_b)+m\sum_{i=1}^{3}\sum_{j=1}^{\infty}\delta_j\ddot{B}_{ij}=Q_{zh} \quad (4b)$$

$$m(\ddot{B}_{ij}+\delta_j\ddot{z}_h)+(EI/L^3)\mu_j^4B_{ij}=Q_{Bij}$$

$$i=1,2,3;\quad j=1,2,\cdots\infty \quad (4c)$$

where Q_{zb},Q_{zh}, and Q_{Bij} are the generalized forces corresponding to z_b,z_h, and B_{ij} degrees of freedom, respectively, arising because of the hydrodynamic forces and internal pressure.

The hydrodynamic forces F_b and F_h acting on the buoy and central head, respectively, are given by[2]

$$F_b=-a_b(\ddot{z}_b-\ddot{z}_w)-(\rho_w/2)C_{db}S_b(\dot{z}_b-\dot{z}_w)\,|\dot{z}_b-\dot{z}_w| \quad (5a)$$

and

$$F_h=-a_h(\ddot{z}_h-\ddot{z}_{wh})-(\rho_w/2)C_{dh}S_h(\dot{z}_h-\dot{z}_{wh})\,|\dot{z}_h-\dot{z}_{wh}| \quad (5b)$$

where a_b,a_h,C_{dh},S_b, and S_h are the corresponding added masses, drag coefficients, and areas of cross section, respectively, and z_{wh} is the wave displacement at the head. The coefficients of added inertia and drag vary somewhat with the size and motion characteristics, but have been assumed to be approximately constant.

The hydrodynamic forces acting on an element $Ld\xi$, located on the ith leg at a distance $L\xi(0\le\xi\le 1)$ from the root, can be written as

$$dF_i=-[a(\ddot{z}_h+\ddot{w}_i-\ddot{z}_{wi})+(\rho_w/2)C_d d(\dot{z}_h+\dot{w}_i-\dot{z}_{wi})\,|\dot{z}_h+\dot{w}_i-\dot{z}_{wi}|L]d\xi \quad (6a)$$

where z_{wi} is the wave displacement at the element and

$$a=\rho_w C_m SL \quad (6b)$$

Realizing that the generalized force Q_j arising because of a set of forces $\bar{F}_k\ (k=1,2,\cdots,n)$ acting at the points $\bar{r}_k\ (k=1,2,\cdots,n)$ is given by[3]

$$Q_j=\sum_{k=1}^{n}\bar{F}_k\cdot\frac{\partial\bar{r}_k}{\partial q_j}$$

one obtains from Eqs. (5) and (6)

$$Q''_{zb}=-a_b(\ddot{z}_b-\ddot{z}_w)-(\rho_w/2)C_{db}S_b(\dot{z}_b-\dot{z}_w)\,|\dot{z}_b-\dot{z}_w| \quad (7a)$$

$$Q''_{zh}=-a_h(\ddot{z}_h-\ddot{z}_{wh})-(\rho_w/2)C_{dh}S_h(\dot{z}_h-\dot{z}_{wh})\,|\dot{z}_h-\dot{z}_{wh}|-a\Big[3\ddot{z}_h-\sum_{i=1}^{3}\int_0^1\ddot{z}_{wi}d\xi+\sum_{i=1}^{3}\sum_{s=1}^{\infty}\delta_s\ddot{B}_{is}\Big]-\left(\frac{\rho_w}{2}\right)C_dLD\sum_{i=1}^{3}\int_0^1\Big\{\dot{z}_h-\dot{z}_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)\dot{B}_{is}\Big\}\,\Big|\dot{z}_h-\dot{z}_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)\dot{B}_{is}\Big|d\xi \quad (7b)$$

and

$$Q''_{Bij}=-a\Big[\delta_j\ddot{z}_h+\ddot{B}_{ij}-\int_0^1\ddot{z}_{wi}\Phi_j(\xi)d\xi\Big]-\left(\frac{\rho_w}{2}\right)C_dLd\int_0^1\Big|\dot{z}_h-\dot{z}_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)\dot{B}_{is}\Big|\Big[\dot{z}_h-\dot{z}_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)\dot{B}_{is}\Big]\Phi_j(\xi)d\xi \quad (7c)$$

where Q''_{zb},Q''_{zh}, and Q''_{Bij} are the generalized forces due to the hydrodynamic forces only.

The contributions of the follower forces to the total generalized forces are given by

$$Q'_{Bij}=\left(\frac{F_a}{L}\right)\sum_{s=1}^{\infty}C_{sj}B_{is} \quad (8a)$$

and

$$Q'_{zb}=Q'_{zh}=0 \quad (8b)$$

where the axial tension F_a is related to the internal pressure by

$$F_a=p\pi d^2/4 \quad (8c)$$

and

$$C_{sj}=\int_0^1\frac{d^2\phi_s}{d\xi^2}\Phi_j d\xi$$

Evaluation of the preceding integral gives[4]

$$C_{sj}=\begin{cases}4(\mu_s\sigma_s-\mu_j\sigma_j)/[(-1)^{s+j}-(\mu_j/\mu_s)^2] & j\neq s\\ \mu_j\sigma_j(2-\mu_j\sigma_j) & j=s\end{cases} \quad (8d)$$

where

$$\sigma_j=(\cosh\mu_j+\cos\mu_j)/(\sinh\mu_j+\sin\mu_j) \quad (8e)$$

Defining

$$\eta_b=z_b/d\quad \eta_h=z_h/d\quad b_{ij}=B_{ij}/d\quad \eta_w=z_w/d$$

$$\eta_{wh}=z_{wh}/d\quad \eta_{wi}=z_{wi}/d\ \text{and}\ \tau=t[c/(m_b+a_b)]^{1/2}$$

the equations of motion, as given by Eq. (4) in conjunction with Eqs. (7) and (8), can be nondimensionalized to yield

$$\eta''_b+(1+\bar{\Omega}^2)\eta_b-\bar{\Omega}^2\eta_h+\alpha_b(\eta'_b-\eta'_w)\,|\eta'_b-\eta'_w|=(\eta_w+f_b\eta''_w) \quad (9a)$$

$$(3+r_{h\ell})\eta''_h+\sum_{i=1}^{3}\sum_{s=1}^{\infty}\delta_s b''_{is}+\bar{\Omega}^2 r_{b\ell}(\eta_h-\eta_b)+\alpha_h(\eta'_h-\eta'_{wh})\,|\eta'_h-\eta'_{wh}|+\alpha\sum_{i=1}^{3}\int_0^1\Big\{\eta'_h-\eta'_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)b'_{is}\Big\}\,\Big|\eta'_h-\eta'_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)b'_{is}\Big|d\xi$$

$$=f_h\eta''_{wh}+f\sum_{i=1}^{3}\int_0^1\eta''_{wi}d\xi \tag{9b}$$

and

$$b''_{ij}+\delta_j\eta''_h+\Omega_j^2 b_{ij}-\bar{P}\sum_{s=1}^{\infty}C_{sj}b_{is}+\alpha\int_0^1\{\eta'_h-\eta'_{wi}$$

$$+\sum_{s=1}^{\infty}\Phi_s(\xi)b'_{is}\}\ |\eta'_h-\eta'_{wi}+\sum_{s=1}^{\infty}\Phi_s(\xi)b'_{is}|\Phi_j(\xi)d\xi$$

$$=f\int_0^1\eta''_{wi}\Phi_j(\xi)d\xi\quad i=1,2,3;\ j=1,2,\cdots\infty \tag{9c}$$

where

$$\bar{\Omega}^2=k/c\qquad \Omega_j^2=\mu_j^4[EI/(m+a)L^3][(m_b+a_b)/c]$$

$$r_{h\ell}=(m_h+a_h)/(m+a)\qquad r_{b\ell}=(m_b+a_b)/(m+a)$$

$$P=F_aL^2/EI\qquad \bar{P}=(F_a/cL)(m_b+a_b)/(m+a)$$

$$\alpha_b=(\rho_w/2)C_{db}S_bd/(m_b+a_b)$$

$$\alpha_h=(\rho_w/2)C_{dh}S_hd/(m+a)\qquad \alpha=2C_d/\pi(1+C_m)$$

$$f_b=a_b/(m_b+a_b)\quad f_h=a_h/(m+a)\quad f=1/(1+C_m) \tag{10}$$

In order to incorporate viscoelastic effects of the legs, the elastic modulus E should be replaced by the corresponding complex modulus $E^*(\omega)=E_1(\omega)+iE_2(\omega)$. The storage modulus $E_1(\omega)$ is associated with the strain energy stored in the body, whereas the loss modulus $E_2(\omega)$ represents the dissipation of energy during cyclic loading.[5] For a viscoelastic solid defined by the three parameters E_1, E_2, and ν_2

$$E^*(\omega)=E_1[1-E_1/(E_1+E_2+i\nu_2\omega)]$$

where E_1 is the instantaneous modulus of the material. Since for polyethylene and mylar $(E_1+E_2)\ll\nu_2\omega$ (unless ω is extremely small),

$$E^*(\omega)\simeq E_1[1+i\omega(E_1/\nu_2\omega^2)]=E_1[1+i\omega\gamma(\omega)]$$

E in the expression for Ω_j^2 in Eq. (10) is replaced by E_1.

III. Vertical Free Vibrations of the System

For the free vibrations of the system, the forcing terms in the equations of motion are equated to zero, i.e.,

$$\eta_w=\eta_{wh}=\eta_{wi}=0$$

Since the effect of the damping terms on the natural frequencies and the modes of the system is of the second order, they may be ignored. If the motion is assumed to be sinusoidal with ω as its dimensionless frequency, Eq. (9) transforms to

$$(1+\bar{\Omega}^2-\omega^2)\eta_b-\bar{\Omega}^2\eta_h=0 \tag{11a}$$

$$-\bar{\Omega}^2r_{b\ell}\eta_b+[\bar{\Omega}^2r_{b\ell}-(3+r_{h\ell})\omega^2]\eta_h-\omega^2\sum_{i=1}^{3}\sum_{s=1}^{\infty}\delta_sb_{is}=0 \tag{11b}$$

and

$$-\omega^2\delta_j\eta_h+(\Omega_j^2-\omega^2)b_{ij}-\bar{P}\sum_{s=1}^{\infty}C_{sj}b_{is}=0$$

$$i=1,2,3;\qquad j=1,2,\cdots\infty \tag{11c}$$

The previous set contains infinite number of equations. In the numerical computations, however, only the first six modes were retained, so that Eq. (11) reduced to an eigenvalue problem of the type

$$[A]\ (x)=\omega^2[B]\ (x) \tag{12a}$$

of order 20. Premultiplying Eq. (12a) with $[B]^{-1}$, one obtains

$$[B]^{-1}\ [A]\ (x)=\omega^2(x)$$

or

$$[C]\ (x)=\omega^2\ (x) \tag{12b}$$

where

$$[C]=[B]^{-1}\ [A]$$

The system of Eqs. (12b) now can be solved by an iteration procedure to obtain the frequencies and mode shapes.

IV. Response of the System to Surface Wave Excitations

The system, under normal operating conditions, will be subjected to the ocean waves which, in general, would lead to both horizontal and vertical motions of the buoy. Obviously, the resulting dynamical analysis of the system indeed will be quite complicated. Fortunately, considerable simplification in the analysis can be achieved without substantially affecting the physics of the problem by examining the system response with the buoy at the crest of a standing wave. Moreover, a complex wave always can be expanded in a Fourier series, and the general forced motion can be obtained by following an approximate analytical procedure similar to the one used in the present analysis for a simple sinusoidal wave.

It can be shown that, for a standing wave,[6]

$$\eta_w=\eta_0\cos2\pi(t/\bar{T})$$

$$\eta_{wh}=\eta_0e^{-2\pi H/L_\lambda}\cos2\pi(t/\bar{T})$$

$$\eta_{wi}=\eta_0e^{-2\pi H/L_\lambda}\cos2\pi(t/\bar{T})\cos2\pi(x/L_\lambda)$$

provided the crest lies along the vertical axis of the system. Here $\eta_0, \bar{T}$, and L_λ are the amplitude, period, and length of the wave, respectively. It may be noticed that the particle motion decreases rapidly with depth. For $H=L_\lambda/2$, the amplitude of particle motion is $\eta_0/23.1$, whereas at a depth equal to the wavelength, the motion reduces to $\eta_0/535$. With the average wavelength of around 100 ft (sea state 3) and cable length of 100-400 ft, it may be assumed that

$$\eta_{wh}=\eta_{wi}\simeq0 \tag{13a}$$

and

$$\eta_w=\eta_0\cos2\pi(t/\bar{T})\equiv\eta_0\cos\omega\tau \tag{13b}$$

where ω is the dimensionless frequency.

The forced motion, in general, will involve all the harmonics of ω; but for simplicity only the fundamental term, which usually is the most important one, is considered. In order to account for the system damping, both sine and cosine terms should be included in the solution. Hence,

$$\eta_b=\eta_{bc}\cos\omega\tau+\eta_{bs}\sin\omega\tau \tag{14a}$$

$$\eta_h=\eta_{hc}\cos\omega\tau+\eta_{hs}\sin\omega\tau \tag{14b}$$

$$b_{ij}=b_{ijc}\cos\omega\tau+b_{ijs}\sin\omega\tau \tag{14c}$$

Substitution of Eq. (14) in Eq. (9) will not, in general, satisfy the equations for all τ; however, one can use Ritz's averaging technique, which involves multiplying both the sides of each equation by $\cos\omega\tau$ and $\sin\omega\tau$ in turn, and integrating over a period. The resulting algebraic equations are

$$(1+\bar{\Omega}^2-\omega^2)\eta_{bc}-\bar{\Omega}^2\eta_{hc}+\alpha_b(8\omega^2/3\pi)\eta_{bs}$$
$$[(\eta_{bc}-\eta_0)^2+\eta_{bs}^2]^{1/2}=(1-f_b\omega^2)\eta_0 \tag{15a}$$

$$(1+\bar{\Omega}^2-\omega^2)\eta_{bs}-\bar{\Omega}^2\eta_{hs}-\alpha_b(\frac{8\omega^2}{3\pi})(\eta_{bc}-\eta_0)$$
$$\times[(\eta_{bc}-\eta_0)^2+\eta_{bs}^2]^{1/2}=0 \tag{15b}$$

$$-r_{b\ell}\bar{\Omega}^2\eta_{bc}+[r_{b\ell}\bar{\Omega}^2-(3+r_{h\ell})\omega^2]\eta_{hc}-\omega^2$$
$$\times\sum_{i=1}^{3}\sum_{k=1}^{\infty}\delta_k b_{ikc}+\alpha_h(\frac{8\omega^2}{3\pi})\eta_{hs}$$
$$(\eta_{hc}^2+\eta_{hs}^2)^{1/2}+\alpha(\frac{8\omega^2}{3\pi})\sum_{i=1}^{3}\int_0^1 D_s(D_c^2+D_s^2)^{1/2}d\xi=0 \tag{15c}$$

$$-r_{b\ell}\bar{\Omega}^2\eta_{bs}+[r_{b\ell}\bar{\Omega}^2-(3+r_{h\ell})\omega^2]\eta_{hs}-\omega^2$$
$$\times\sum_{i=1}^{3}\sum_{k=1}^{\infty}\delta_k b_{iks}-\alpha_h(\frac{8\omega^2}{3\pi})$$
$$\eta_{hc}(\eta_{hc}^2+\eta_{hs}^2)^{1/2}-\alpha(\frac{8\omega^2}{3\pi})\sum_{i=1}^{3}\int_0^1 D_c(D_c^2+D_s^2)^{1/2}d\xi=0 \tag{15d}$$

$$(\Omega_j^2-\omega^2)b_{ijc}+\gamma\omega\Omega_j^2 b_{ijs}-\omega^2\delta_j\eta_{hc}-\bar{P}$$
$$\times\sum_{k=1}^{\infty}C_{kj}b_{ikc}+\alpha(\frac{8\omega^2}{3\pi})$$
$$\int_0^1\Phi_j(\xi)D_s(D_c^2+D_s^2)^{1/2}d\xi=0 \tag{15e}$$

$$-\gamma\omega\Omega_j^2 b_{ijc}+(\Omega_j^2-\omega^2)b_{ijs}-\omega^2\delta_j\eta_{hs}-\bar{P}$$
$$\times\sum_{k=1}^{\infty}C_{kj}b_{iks}-\alpha(\frac{8\omega^2}{3\pi})$$
$$\int_0^1\Phi_j(\xi)D_c(D_c^2+D_s^2)^{1/2}d\xi=0$$
$$i=1,2,3 \quad k=1,2,\cdots \tag{15f}$$

where

$$D_c=\eta_{hc}+\sum_{k=1}^{\infty}\Phi_k(\xi)b_{ikc}$$

$$D_s=\eta_{hs}+\sum_{k=1}^{\infty}\Phi_k(\xi)b_{iks}$$

and γ the equivalent viscoelastic damping.

The solution of these simultaneous equations gives the sine and cosine components of each generalized coordinate. Since the first few modes are likely to be the most important ones, only the first two of the set of Eqs. (15f) are considered in the numerical computations. This effectively includes the first four natural frequencies of the coupled system.

V. Results and Discussion

A. Free Vibration

By truncating the infinite order system to the first m modes, $(3m+2)$ eigenvalues are obtained from Eq. (11). Two identical sets of m eigenvalues resulted, along with a third set containing $(m+2)$ frequencies. The repeated eigenvalues correspond to the independent motion of the cantilevers at their natural frequencies while the buoy and the central body are at rest. On the other hand, the nonrepeated eigenvalues described the coupled motion in which all of the legs moved identically, and hence only $(m+2)$ eigenvalues can correspond to this type of motion. This can be explained in the light of the restrictions on the problem. Since only pure vertical motions are considered, the sum of the shear forces at the root for coupled motion must be nonzero (equal to the inertia force of the central head), whereas all of the components of the resultant moment must vanish. It appears that only identical

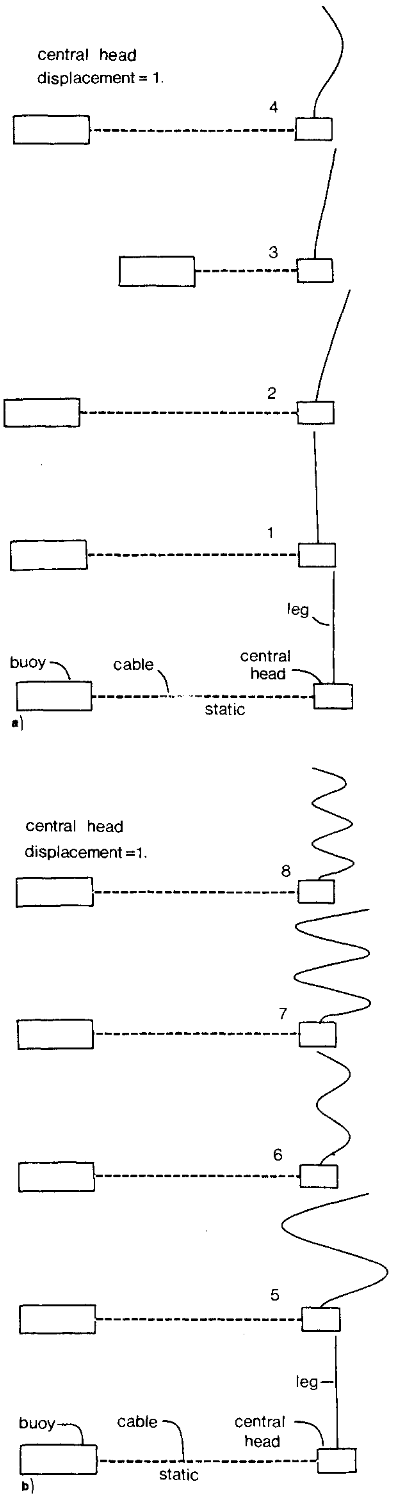

Fig. 3 Modes of coupled vertical motion a) $i=1$ to 4 b) $i=5$ to 8.

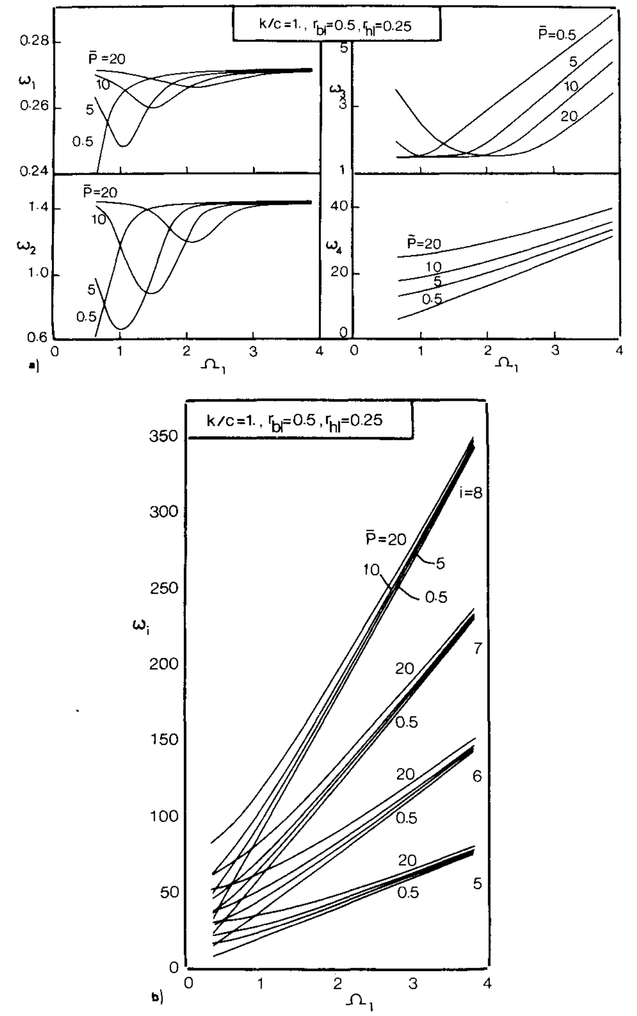

Fig. 4 Variation of natural frequencies of coupled vertical motion with the pressure parameter and dimensionless fundamental leg frequency a) $i = 1$ to 4 b) $i = 5$ to 8.

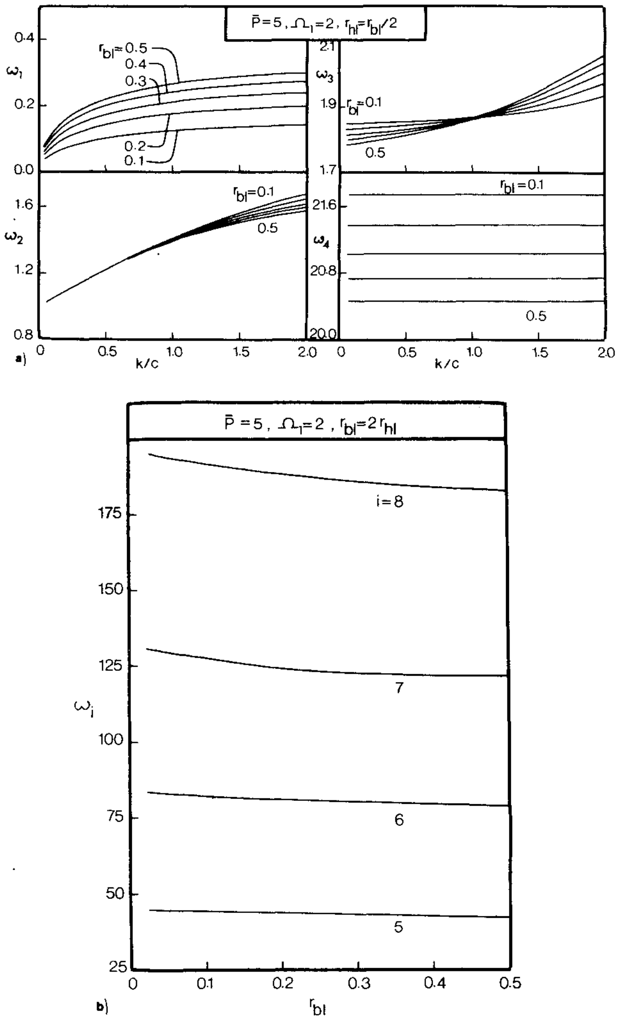

Fig. 5 Variation of natural frequencies of coupled vertical motion with the spring stiffness and weight of the head a) $i = 1$ to 4 b) $i = 5$ to 8.

leg motions satisfy all these requirements. In the present case, m is taken to be 6.

The typical amplitudes of motion of the buoy, central head, and the cylindrical legs during coupled motion at fundamental and higher natural frequencies are shown in Fig. 3. It may be pointed out that, in order to emphasize the relative motion, only the displacements, are presented to the scale (unit central head displacement), geometrical dimensions being left arbitrary for clarity. One may notice that, for the lowest three frequencies, the shape of the cylinder resembles its fundamental mode since it is the dominant one. However, the subsequent frequencies correspond to the second and higher modes of the leg. Since the third natural frequency of the coupled motion is quite close to the natural frequency of the buoy due to is buoyancy, the buoy has a large displacement.

The variation of the coupled natural frequencies with $\bar{P}$ and Ω_l for given $\bar{\Omega}, r_{h\ell}$, and $r_{b\ell}$ is shown in Fig. 4. Here $\bar{P}$ characterizes the effect of the axial tension, whereas Ω_l is the fundamental frequency of each leg. The first three of these frequencies, for a given $\bar{P}$, first decrease and then increase with increasing Ω_l (Fig. 4a). For large values of Ω_l, these frequencies decrease with increasing $\bar{P}$, whereas for small values of the fundamental frequency, the behavior is exactly the opposite. This is so because for large Ω_l the structure behaves like a cantilever, whereas for small Ω_l it acts as a string. It is well-known that increasing the tensile follower force decreases the fundamental frequency of a cantilever,[7] but this has an opposite effect on a string. The behavior of the fourth (Fig. 4a) and higher frequencies (Fig. 4b) is the same as that of the second and higher frequencies of a single cylinder, i.e., they increase with Ω_l and $\bar{P}$. If Ω_l is not too small, they vary linearly.

The variation of the coupled frequencies with $\bar{\Omega}^2$ and $r_{b\ell}$ for given $\bar{P}$, Ω_l, and $r_{h\ell}/r_{b\ell}$ is plotted in Fig. 5. The first three increase with $\bar{\Omega}$, i.e., the stiffness of the spring, whereas the subsequent ones are almost independent of it. The parameter $r_{b\ell}$, representing the ratio of the apparent masses of the buoy and the leg, has opposite effects on the lower and higher frequencies. The higher frequencies (Fig. 5b), which are characterized by the stiffness of the legs, decrease slightly with $r_{b\ell}$, whereas the lowest one, which involves large coupling between the buoy and the array, increases with the same parameter.

Given the operating sea conditions, the parameters must be so chosen as to yield the natural frequencies of the system (at least the lower ones) far removed from the forcing frequencies.

B. Forced Vibration

The frequency response of the buoy, central head, and the tip of a leg, for different $\bar{\Omega}^2 (= k/c)$, are plotted in Fig. 6a. One may notice that the buoy displacement peaks at smaller frequencies with reduction in k/c. For $k/c = 1$, there is a less conspicuous peak, since around this value of k/c the array acts somewhat like a dynamic absorber. For the motions of the central head and leg tips, resonance is observed first at a very small frequency (fundamental) and subsequently at higher frequencies. It may be observed that these resonant displacements diminish with k/c, i.e., if the elastic cable is a

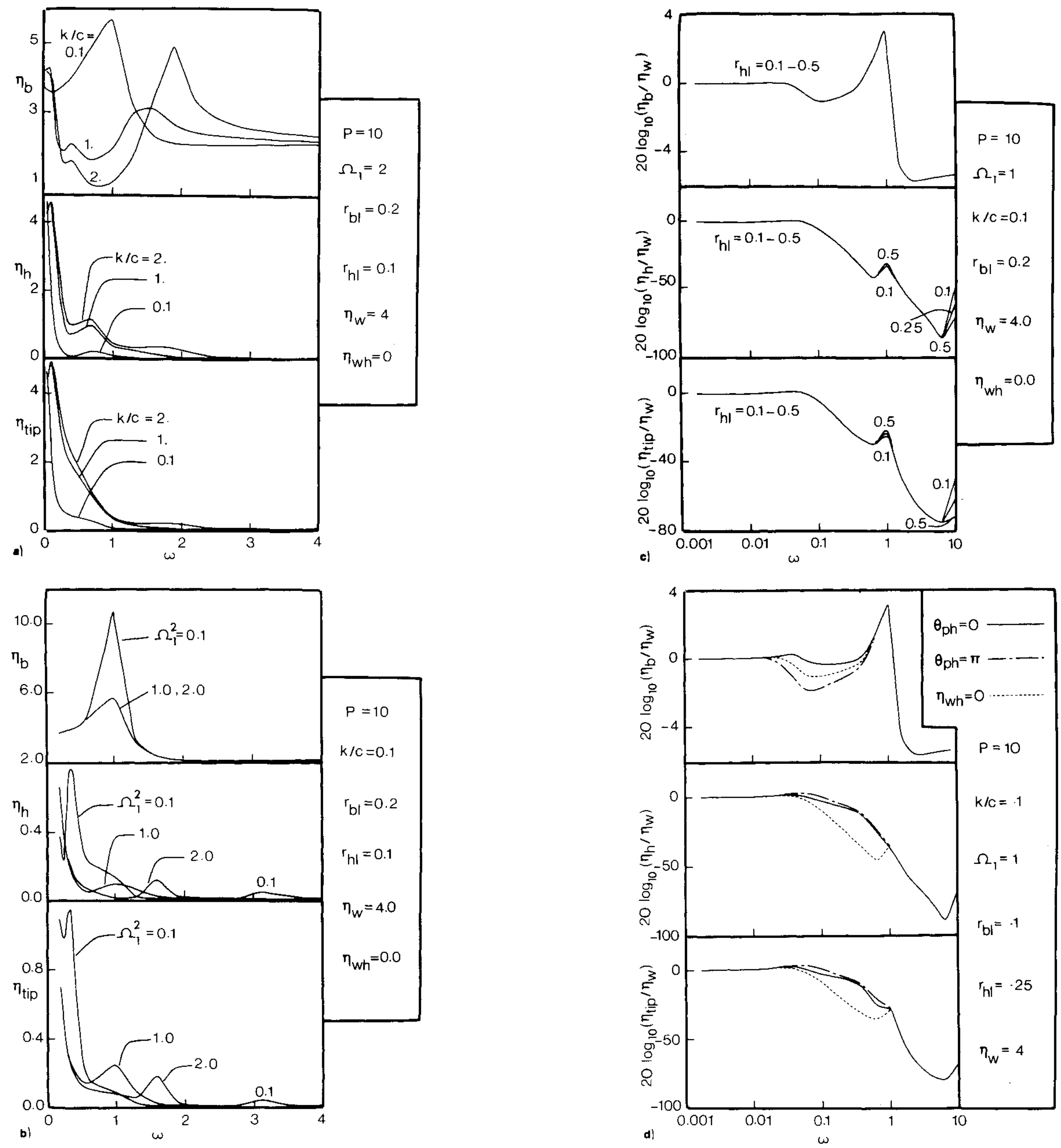

Fig. 6 Frequency response of the buoy, central head, and the tip of a leg as affected by the following: a) Equivalent spring stiffness b) Fundamental frequency of a leg c) Weight of the central head d) Wave amplitude at the central head.

soft spring, the motion at the water surface is not transmitted to the array.

Figure 6b shows the frequency response when Ω_l is varied. It is evident that the displacements of the tips of the legs could be reduced by increasing Ω_l, i.e., by making the legs more stiff. This implies reduction in length of the legs and increase in their diameter, thickness, and elastic modulus. Moreover, it may be noted from Fig. 6c (in logarithmic scales) that, for a given Ω_l, $\bar{P}$, and $\bar{\Omega}$, the tip displacement for higher forcing frequencies diminishes with increasing $r_{h\ell}$, whereas that at lower frequencies remains unaffected.

The effect of taking the wave displacements η_{wh} and η_{wi} into account is indicated in Fig. 6d. Here η_{wh} and η_{wi} are assumed to be equal and have a constant phase difference θ_{ph} with respect to the surface wave displacement η_w. Clearly, consideration of η_{wh} and η_{wi} increases the displacements of the central head and tip of each leg for moderate forcing frequencies; $\theta_{ph} = \pi$ represents a more adverse situation than $\theta_{ph} = 0$.

VI. Conclusions

The important conclusions based on the analysis can be summarized as follows:

1) The solution of the eigenvalue problem for free vertical oscillation of the buoy-cable-array system yields two sets of repeated natural frequencies corresponding to the independent motion of the legs, and a third set describing the coupled motions. All the three legs move identically during the coupled pure vertical oscillations.

2) The variation of the natural frequencies with different system paramters, as obtained in this study, should prove useful in a design procedure aimed at avoiding resonance.

3) Analysis of the response of the system to surface wave excitations suggests that the displacements of the leg tips can be reduced by use of an elastic cable with small stiffness and legs having a large fundamental frequency. The typical value of this frequency as observed in the prototype structures is below 2 cps. The analysis suggests that any increase in this value is likely to have beneficial influence on the structural

response. On the other hand, as emphasized by Fig. 6b, very small values of leg frequency may lead the buoy to leave the water surface, and hence must be avoided.

4) Although an increase in the inertia of the central head is likely to reduce tip deflections, it would be difficult to realize this from design considerations.

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